Martingales in Homogeneous spaces

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Abstract

Let G/H be a reductive homogeneous space and $\nabla^{G/H}$ a G-invariant connection. Our interesse is to study $\nabla^{G/H}$-martingales in G/H. In fact, we yields a correspondence between $\nabla^{G/H}$-martingales and local martingales $\mathfrak{m}$, where $\mathfrak{m}$ is the subspace of Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $Ad(H)(\mathfrak{m}) \subset \mathfrak{m}$. Here $\mathfrak{h}$ is the Lie subalgebra of H. As application we show that martingales in the sphere S^n are in 1-1 correspondence with local martingales in $\mathbb{R}^n$.

Key words: Homogeneous space; martingales; stochastic analysis on manifolds

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1 Introduction

Let G be a Lie Group and H closed Lie subgroup. In this work we consider the reductive homogeneous spaces. It means that $\mathfrak{g}, \mathfrak{h}$ are Lie algebras of G and H, respectively, and there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad(H)(\mathfrak{m}) \subset \mathfrak{m}$. Our intention is to study the martingales in G/H with respect to G-invariant connections. A first study in this direction was done by M. Arnaudon in [3], where he characterized the martingales with respect the canonical connection in G/H in function of local martingales in $\mathfrak{m}$. The reader can be see that his strategy was used the stochastic exponential in the sense of Stratonovich (see for example [8]) to show this.

In our paper, being natural to see $\pi : G \to G/H$ as submersion, furthermore, as principal fiber bundle, our idea is given a G-invariant connection $\nabla^{G/H}$ on G/H and to construct a desirable connection ∇^G on G such that $\pi : G \to G/H$ is an affine submersion with horizontal distribution. It means that $\pi*(\nabla^G_{A^h} B^h) = \nabla^{G/H}_A B$, where X, Y are vector fields on G/H and A^h, B^h are their lifts to G, respectively. The last definition was introduced by N. Abe and K. Hasewaga in [1].

Take the connections $\nabla^{G/H}$ and ∇^G as above. Following the natural idea of projecting the horizontal geodesics of G in geodesics of G/H we wish to project horizontal ∇^G-martingales in $\nabla^{G/H}$-martingales. To make the role of geodesics in G we will use the Itô exponential on G, which was introduced by author in [15]. Given a local martingale M in $\mathfrak{g}$ the Itô exponential $X = e^G(M)$ with respect to ∇^G is the solution of the stochastic differential equation in Itô sense:

$$d^{\nabla^G} X_t = L_{(X_t)*}(e)dM, \quad X_0 = e.$$

In context proposed until here, our main Theorem says:

Theorem : *Let G/H a reductive homogeneous space G/H. Let $\nabla^{G/H}$ and ∇^{G} connections on G/H and G, respectively, such that π is an affine submersion with horizontal distribution. If X_t is a $\nabla^{G/H}$-martingale in G/H, then it is written as $\pi \circ e^{G}(M)$, where M is a local martingale in $\mathfrak{m}$.*

The hypothesis of Theorem is satisfied in many examples of homogenous spaces, which we give in this work. However, a special application is the sphere. Viewing the sphere S^n as homogeneous space we show that the martingales in sphere are in 1-1 correspondence with local martingales in $\mathbb{R}^n$.

2 Stochastic calculus

In this work we use freely the concepts and notations of P. Protter [12], P. Meyer [10], M. Emery [6] and [7], S. Kobayashi and N. Nomizu [9] and J. Cheeger and D.G. Ebin [5]. We suggest the reading of [4] for a complete survey about the objects of this section. From now on the adjective smooth means C^{∞}.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for example [6]). Our basic assumption is that every stochastic process is continuous.

Let M be a smooth manifold and X_t a continuous stochastic process with values in M. We call X_t a semimartingale if, for all f smooth function, $f(X_t)$ is a real semimartingale.

Let M be a smooth manifold with connection ∇^M. Let X be a continuous semimartingale with values in M, θ a section of TM^* and b a section of $T^{(2,0)}M$. We denote by $\int \theta d^{\nabla} X$ the Itô integral of θ along X and by $\int b \; d(X, X)$ the quadratic integral of b along X. We recall that X is a ∇-martingale if and only if $\int \theta \; d^{\nabla^M} X$ is a local martingale for any $\theta \in \Gamma(TM^*)$.

Let M and N be smooth manifolds endowed with connections ∇^M and ∇^N, respectively, and $F : M \to N$ a smooth map. P. Catuogno in [4] shows the following version for Itô formula in smooth manifolds, which will be said geometric Itô formula:

$$\int_0^t \theta \; d^N F(X) = \int_0^t F^*\theta \; d^M X + \frac{1}{2}\int_0^t \beta_F^*\theta \; (dX, dX), \tag{1}$$

where β_F is the second fundamental form of F and $\theta \in \Gamma(T^*N)$.

From the above formula, it follows that F is an affine map if it and only if sends ∇^M-martingales to ∇^N-martingales.

3 Connections on homogeneous spaces

Let H be a closed Lie subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. We assume that the homogeneous space G/H is reductive, that is, there is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad(H)(\mathfrak{m}) \subset \mathfrak{m}$. Let π be the natural mapping of G onto the space G/H of the cosets gH, $g \in G$. Also, for each $a \in G$ we define $\tau_a : G/H \to G/H$ by $\tau_a(gH) = agH$, the left translation. If $a \in G$ and L_a are the left translation on G, then

$$\pi \circ L_a = \tau_a \circ \pi.$$

The differential of π at e shows that $\ker(d\pi)_e = \mathfrak{h}$. Since $d\pi$ is onto we get the canonical isomorphism $\mathfrak{m} \cong T_o(G/H)$.

As the left translation L_g is a diffeomorphism, for every $g \in G$, we have

$$T_g G = (L_g)_{*e}\mathfrak{h} \oplus (L_g)_{*e}\mathfrak{m}.$$

Thus, writing

$$TG_{\mathfrak{h}} := \{(L_g)_{*e}\mathfrak{h}; \forall g \in G\} \quad \text{and} \quad TG_{\mathfrak{m}} := \{(L_g)_{*e}\mathfrak{m}; \forall g \in G\}$$

follows that $TG = TG_{\mathfrak{h}} \oplus TG_{\mathfrak{m}}$.

Let us denote the Maurer-Cartan form on G as ω. Theorem 11.1 in [9] shows that the principal fiber bundle $G(G/H, H)$ has the vertical part of the Maurer-Cartan as a connection form with respect to decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In other words, $TG_{\mathfrak{m}}$ is a connection in $G(G/H, H)$. The horizontal lift from G/H to G is denoted by $\mathcal{H}$ and the horizontal projection of TG into $TG_{\mathfrak{m}}$ is written as $\mathbf{h}$.

Let $A \in \mathfrak{m}$. The left invariant vector field $\tilde{A}$ on G is denoted by $\tilde{A}(g) = L_{g*}A$ and the G-invariant vector field A_* on G/H is defined by $A_* = \tau_{g*}A$. It is clear that $\tilde{A}$ is a horizontal vector field on G.

It is well-known, see Theorem 8.1 in [11], that for each G-invariant connection $\nabla^{G/H}$ is associated to a unique $Ad(H)$-invariant bilinear map $\beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, that is,

$$\beta(Ad(H)(A), Ad(H)(B)) = Ad(H)\beta(A, B), \quad A, B \in \mathfrak{m}.$$

This correspondence is given by

$$(\nabla^{G/H}_A B)_o = \beta(A, B), \quad A, B \in \mathfrak{m}.$$

Since we are interested on martingales in G/H, our idea is choose a good connection ∇^G such that it is horizontally projected over $\nabla^{G/H}$. In other words, we choose ∇^G in the way that $\pi : G \to G/H$ is an affine submersion with horizontal distribution. This definition was given by N. Abe and H. Hasegawa in [1] and it means the following. Taking $A, B \in \mathfrak{m}$ we yields the left invariant vectors fields $\tilde{A}, \tilde{B}$ on G and the G-invariant vector fields A_*, B_* on G/H. It is clear that $\tilde{A}, \tilde{B}$ are horizontal and $\pi_*(\tilde{A}) = A_*$ and $\pi_*(\tilde{B}) = B_*$. In other words, $\tilde{A}, A_*$ and $\tilde{B}, B_*$ are π- related. Furthermore, $\tilde{A}, \tilde{B}$ are horizontal lift of A_*, B_*, respectively. Therefore π is an affine submersion with horizontal distribution if

$$\mathbf{h}(\nabla^G_{\tilde{A}}\tilde{B}) = \mathcal{H}(\nabla^{G/H}_{A_*}B_*).$$

A natural way to construct a connection ∇^G from $\nabla^{G/H}$ such that π is affine submersion with horizontal distribution is to extend β to a bilinear map α to $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ such that $\alpha(A, B) = \beta(A, B)$ for $A, B \in \mathfrak{m}$. Thus, there exists a left invariant connection ∇^G on G such that

$$(\nabla^G_{\tilde{A}}\tilde{B})(e) = \alpha(A, B), \quad X, Y \in \mathfrak{g}.$$

We prove some geometric necessary facts.

Proposition 3.1 *Let $\nabla^{G/H}, \nabla^G$ be connections such that π is an affine submersion with horizontal distribution.*

1. *If $f \in C^\infty(G/K)$ then*

$$Hess^G(f\circ\pi)(g)(\tilde{A},\tilde{B}) = Hess^{G/H}(f)(\pi(g))(A_*,B_*),$$

for $A,B\in\mathfrak{m}$.

2. *If $A,B\in\mathfrak{m}$ then*

$$\beta_\pi(\tilde{A},\tilde{B}) = 0,$$

where β_π is the second fundamental form of π.

Proof: *1.* For $A,B\in\mathfrak{m}$, $\pi_*(g)(\tilde{A}(g)) = A_*(\pi(g))$ and $\pi_*(g)(\tilde{B}(g)) = B_*(\pi(g))$ for all $g\in G$. By definition of hessiano, for every $f\in C^\infty(G/K)$,

$$\begin{aligned} Hess^{G/H}(f(\pi(g)))(A_*,B_*) &= A_*(\pi(g))(B_*f) - df(\nabla^{G/H}_{A_*}B_*)(\pi(g)) \\ &= \tilde{A}(g)\tilde{B}(f\circ\pi) - (f\circ\pi)_*(\nabla^G_{\tilde{A}}\tilde{B})(g) \\ &= Hess^G(f\circ\pi)(g)(\tilde{A},\tilde{B}). \end{aligned}$$

2. Given $A,B\in\mathfrak{m}$ we have, by definition of the second fundamental form,

$$\beta_\pi(\tilde{A},\tilde{B}) = \nabla^{G/H}_{\pi_*\tilde{A}}\pi_*\tilde{B} - \pi_*\nabla^G_{\tilde{A}}\tilde{B}.$$

Being π an affine submersion with horizontal distribution, we obtain

$$\beta_\pi(\tilde{A},\tilde{B}) = \nabla^{G/H}_{A_*}B_* - \nabla^{G/H}_{A_*}B_* = 0.$$

4 Martingales in homogeneous space

We endow G with a left invariant connection ∇^G and $\mathfrak{g}$ with a flat connection $\nabla^{\mathfrak{g}}$. In [15], the author defines the Itô stochastic exponential with respect to ∇^G and $\nabla^{\mathfrak{g}}$ as the solution of the Itô stochastic differential equation

$$d^{\nabla^G}X_t = L_{(X_t)*}(e)dM, \quad X_0 = e, \tag{2}$$

where M is a semimartingale in $\mathfrak{g}$. For simplicity, we call $e^G(M)$ of Itô exponential. In [15], we have the following results about Itô exponential

Theorem 4.1 *Given a semimartingale X in G, there exists a unique semimartingale M in $\mathfrak{g}$ such that $X = e^G(M)$.*

Theorem 4.2 *Let ∇^G be a connection on G. The ∇^G-martingale in G are exactly the process $e^G(M)$ where M is a local martingale on $\mathfrak{g}$.*

Before we work with martingales in G/H it is necessary to develop a result in the Lie group G. It is related with the left translate of semimartingales by a random variable with values in G. In consequence, we see that the set of martingales in G with respect to a left invariant connection do not change if we translate it to left by a random variable with values in G.

Proposition 4.3 *Let G be a Lie group and ∇^G a left-invariant connection on G. If Y_t is a semimartingale on G and ξ is a random variable with values in G, then, for θ 1-form on G,*

$$\int \theta d^{\nabla^G}\xi Y_t = \int (L_\xi^*\theta)d^{\nabla^G}Y_t.$$

Proof: We begin denoting the product on Lie group G by m. Let θ be a 1-form on G. As a function to m, the Itô integral along ξY_t is writing as

$$\int \theta d^{\nabla^G} \xi Y_t = \int \theta d^{\nabla^G} m(\xi, Y_t).$$

The geometric Itô formula (1) gives

$$\int \theta d^{\nabla^G} \xi Y_t = \int m^* \theta d^{\nabla^G \times \nabla^G} (\xi, Y_t) + \frac{1}{2} \int \beta_m^* \theta (d(\xi, Y_t), d(\xi, Y_t)).$$

From Proposition 3.15 in [7] we see that

$$\int \theta d^{\nabla^G} \xi Y_t = \int (R_{Y_t}^* \theta) d^{\nabla^G} \xi + \int (L_\xi^* \theta) d^{\nabla^G} Y_t + \frac{1}{2} \int \beta_m^* \theta (d(\xi, Y_t), d(\xi, Y_t)).$$

ξ is viewed a constant process, and consequently

$$\int \theta d^{\nabla^G} \xi Y_t = \int (L_\xi^* \theta) d^{\nabla^G} Y_t + \frac{1}{2} \int \beta_m^* \theta (d(\xi, Y_t), d(\xi, Y_t)).$$

We claim that the $\beta_m(d(\xi, Y_t), d(\xi, Y_t))$ is null. In fact, let $0 \in T_g G$ and Y a left invariant vector field on G. Here, 0 is the vector associated to the constant process ξ. Then

$$\begin{aligned}
\beta_m(0, Y) &= \nabla^G_{m_*(0,Y)} m_*(0, Y) - m_* \nabla^{G \times G}(0, Y) \\
&= \nabla^G_{R_{h*}0 + L_{g*}(Y)} (R_{h*}0 + L_{g*}(Y)) - m_* \nabla^{G \times G}(0, Y) \\
&= \nabla^G_{L_{g*}Y} L_{g*} Y - L_{g*}(\nabla^G_Y Y) \\
&= L_{g*}(\nabla^G_Y Y) - L_{g*}(\nabla^G_Y Y) \\
&= 0,
\end{aligned}$$

where in forth equality we use the fact that ∇^G is a left invariant connection. Thus we get

$$\int \theta d^{\nabla^G} \xi Y_t = \int (L_\xi^* \theta) d^{\nabla^G} Y_t.$$

Corollary 4.4 *Let G be a Lie group and ∇^G a left-invariant connection on G. Let ξ be a random variable with values in G. A semimartingale Y_t in G is ∇^G-martingale if and only if ξY_t so is.*

The way used to work with martingales in G/H is to see $\pi : G \to G/H$ as G-principal fiber bundle and to make use of the horizontal lift of semimartingale due to I. Shigegawa in [14]. The horizontal lift in our context is expressed as: if X_t is a $\nabla^{G/H}$-martingale in G/H, it is clear that X_t is a semimartingale in G/H. As $\pi : G \to G/H$ is a H-principal fiber bundle there is a unique horizontal lifting Y_t in G such that $\pi(Y_t) = X_t$ and $\int \omega \delta X_t = 0$ (see Theorem 2.1 in [14]), where ω is the vertical part of Maurer-Cartan form associated with horizontal distribution $TG_{\mathfrak{m}}$.

Proposition 4.5 *Let G/H a reductive homogeneous space G/H. Let $\nabla^{G/H}, \nabla^G$ be connections such that π is an affine submersion with horizontal distribution. If X_t is a $\nabla^{G/H}$-martingale in G/H such that $X_0 = \pi(Y_0)$, where Y_0 is random variable in G, then so is $Z_t = \tau_{Y_0^{-1}} X_t$.*

Proof: Let X_t be a $\nabla^{G/H}$-martingale and Y_t its horizontal lift to G. Taking a 1-form θ on G/H follows

$$\int \theta d^{G/H} Z_t = \int \theta d^{G/H} \tau_{Y_0^{-1}} X_t = \int \theta d^{G/H} \tau_{Y_0^{-1}} \pi(Y_t) = \int \theta d^{G/H} \pi(L_{Y_0^{-1}} Y_t).$$

From the geometric Itô formula (1) and Proposition 3.1 we see that

$$\begin{aligned} \int \theta d^{G/H} Z_t &= \int \pi^* \theta d^{G/H} (L_{Y_0^{-1}} Y_t) + \int \pi_* \theta \beta_\pi (d(L_{Y_0^{-1}} Y_t), d(L_{Y_0^{-1}} Y_t)) \\ &= \int \pi^* \theta d^G (L_{Y_0^{-1}} Y_t). \end{aligned}$$

Proposition 4.3 now assures that

$$\int \theta d^{G/H} Z_t = \int \theta \pi^* L^*_{Y_0^{-1}} d^G Y_t = \int \theta \tau^*_{Y_0^{-1}} \pi^* d^G Y_t.$$

Again, from geometric Itô formula (1) and Proposition 3.1 we conclude that

$$\int \theta d^{G/H} Z_t = \int \theta \tau^*_{Y_0^{-1}} d^{G/H} \pi(Y_t) = \int \theta \tau^*_{Y_0^{-1}} d^{G/H} X_t.$$

Since X_t is $\nabla^{G/H}$-martingale, it follows that Z_t is a $\nabla^{G/H}$-martingale.

Proposition above allows considering $\nabla^{G/H}$-martingales with initial condition o, that is, we can consider only the $\nabla^{G/H}$-martingales X_t with $X_0 = o$, where $o = H$ is the origin in G/H.

Lemma 4.6 *Let G/H a reductive homogeneous space G/H. Let $\nabla^{G/H}$ and ∇^G connections on G/H and G, respectively, such that π is an affine submersion with horizontal distribution. If U_t is a horizontal parallel stochastic transport along X_t, then $\pi_*(U_t)$ is a parallel stochastic transport along the semimartingale $\pi(X_t)$ in G/K.*

Proof: It is sufficient to show that $\pi_*(U_t)$ satisfies the formula of the parallel stochastic transport, see for instance (8.11) in [6]. Consider $f \in C^\infty(G/K)$. Applying this formula we obtain that

$$\begin{aligned} (\pi_* U_t) f + (\pi_* U_0) f &= U_t(f \circ \pi) + U_0(f \circ \pi_*) \\ &= \int Hess(f \circ \pi)(U_t, \delta X_t) \\ &= \int Hess(f)(\pi_* U_t, \delta \pi(X_t)), \end{aligned}$$

where we used the Proposition 3.1 in the later equality. It follows immediately that $\pi_*(U_t)$ is parallel stochastic transport along $\pi(X_t)$.

Theorem 4.7 *Let G/H a reductive homogeneous space G/H. Let $\nabla^{G/H}$ and ∇^G connections on G/H and G, respectively, such that π is an affine submersion with horizontal distribution. If X_t is a $\nabla^{G/H}$-martingale in G/H, then it is written as $\pi \circ e^G(M)$, where M is a local martingale in $\mathfrak{m}$.*

Proof: Let X_t be a $\nabla^{G/H}$-martingale in G/H and Y_t its horizontal lift in G. Consider a 1-form θ in $T^*(G/K)$. Since π is an affine submersion with horizontal distribution, from Proposition 3.1 and the geometric Itô formula (1) we obtain

$$\int \theta d^{G/H} X_t \quad = \quad \int \theta d^{G/H} \pi(Y_t) = \int (\pi^*\theta) d^G Y_t = \int \theta \pi_* d^G Y_t,$$

where we used that Y_t is a horizontal semimartingale in G. Hence

$$d^{G/H} X_t = \pi_* d^G Y_t.$$

Let $\{H_1, \ldots, H_n\}$ be a basis on $\mathfrak{g}$. Choose $\{H_\kappa, \kappa = 1, \ldots, r\}$ such that it is a basis of $\mathfrak{m}$. By Theorem 4.1, there is a unique semimartingale N in $\mathfrak{g}$ such that $d^G Y_t = L_{Y_t *} dN$. If we write $N = \sum_{\kappa=1}^{r} N^\kappa H_\kappa + \sum_{j=r+1}^{n} N^j H_j$, then $d^G Y_t = dN^\kappa U_t^\kappa + dN^j U_t^j$, where $U_t^i = L_{Y_t *} H_i, i = 1, \ldots, n$. It is obvious that $\sum_{\kappa=1}^{r} N^\kappa H_\kappa$ is a semimartingale in $\mathfrak{m}$ and that

$$d^{G/H} X_t = \pi_*(dN_t^\kappa U_t^\kappa) = dN_t^\kappa \pi_*(U_t^\kappa). \tag{3}$$

The set $\{U^1, \ldots, U^n\}$ is a moving frame along Y_t (see [6] for the definition of moving frame). Hence $\{\pi_*(U^1), \ldots, \pi_*(U^r)\}$ is a moving frame along X_t, by Lemma above. Let us denote by $\{\eta_1, \ldots, \eta_r\}$ the dual basis of $\{\pi_*(U_t^\kappa), \kappa = 1, \ldots, r\}$ along X_t. Define $M_t = \sum_{l=1}^{r} M_t^l H_l$ a semimartingale in $\mathfrak{m}$, where $M_t^l = \int \eta_l d^{G/H} X_t$. For every $l = 1, \ldots, r$, we claim that $M_t^l = N_t^l$. In fact,

$$M_t^l = \int \eta_l d^{G/H} X_t = \int \eta_l dN_t^\kappa \pi_*(U_t^\kappa) = \int dN_t^\kappa \eta_l \pi^* U_t^\kappa = \int dN_t^l = N_t^l.$$

It follows that $N_t = M_t + \sum_{l=r+1}^{n} N^l H_l$. From this and (3) we conclude that $d^{G/H} X_t = \pi_*(L_{Y_t *} dM_t)$, and also that

$$d^{G/H} X_t = \tau_{Y_t *} dM_t. \tag{4}$$

The semimartingale M_t above is called the lifting of X_t in $\mathfrak{m}$ (see [6] for this definition). From the stochastic differential equation (4) we conclude directly that X_t is a $\nabla^{G/H}$-martingale if, and only if, M_t is a local martingale in $\mathfrak{m}$. Theorem is proved if we see that $Y_t = e^G(M)$.

Remark 1 In the proof of the Theorem above, we founded a semimartingale $Y_t = e^G(M_t)$. Since M is a local martingale in $\mathfrak{m}$, we can consider M as local martingale in $\mathfrak{g}$. Therefore Y_t is a ∇^G-martingale, which follows from Theorem 4.2. Furthermore, in terms of theory of connections, Y_t can be consider as a horizontal martingale in G.

Remark 2 From the proof of Theorem 4.7 we have that a semimartingale X_t in G/H satisfies the Itô stochastic differential equation

$$d^{G/H} X_t = \tau_{Y_t *} dM_t, X_0 = o, \tag{5}$$

where M_t is a semimartingale in $\mathfrak{m}$ and $o = H$.

Example 4.1 *K. Nomizu in [11] defined by canonical affine connection of the second kind the connection $\nabla^{G/H}$ which has the connection function $\beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ given by $\beta(A, B) = 0$, for $A, B \in \mathfrak{m}$. We extend β for a connection function $\alpha(A, B) = 0$, for $A, B \in g$. Then, the connection ∇^G is given by $\nabla^G_{\tilde{A}} \tilde{B} = 0$. With these connections, it is clear that $\pi : G \to G/H$ is an affine submersion with horizontal distribution. Theorem 4.7 assures that for each $\nabla^{G/H}$-martingale X there exists a local martingale in $\mathfrak{m}$ such that $X_t = \pi \circ e^G(M)$. This result was first proved by M. Arnaudon in [3]. As a particular case of this example we have the Symmetric Spaces which admits a G-invariant metrics (see Theorem 3.3, chapter XI, in [9]).*

Example 4.2 *K. Nomizu in [11] called the canonical affine connection of the first kind the connection $\nabla^{G/H}$ which has the connection function $\beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defined as $\beta(A, B) = \frac{1}{2}[A, B]_{\mathfrak{m}}$. The natural way to extend β to α is to take $\alpha(A, B) = \frac{1}{2}[A, B]$, for $A, B \in \mathfrak{g}$. In according to correspondence between connections on G/H and G and connections functions β and α, respectively, $\nabla^{G/H}_A B = \frac{1}{2}[A, B]_{\mathfrak{m}}$ and $\nabla^G_A B = \frac{1}{2}[A, B]$. It follows directly that $\pi : G \to G/H$ is an affine submersion with distribution horizontal. Therefore every $\nabla^{G/H}$-martingale X_t is written as $X_t = \pi \circ e^G(M_t)$, where M_t is a local martingale in $\mathfrak{m}$, which follows from Theorem 4.7.*

Example 4.3 *A class of homogeneous space that satisfy the Example above are the normal homogeneous spaces. Following Definition 6.60 in [13], a Riemannian homogeneous space $M = G/H$ is called normal homogeneous if there exists a bi-invariant metric on G such that $\pi_*|_e$ maps the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathfrak{g}$ isometrically to $M_{\pi(e)}$. It is know that Levi-Civita connection on G is given by $\nabla^G_A B = \frac{1}{2}[A, B]$, for A, B. In the other side, it is possible to show that the Levi-Civita connection on G/H is given by $\nabla^{G/H}_A B = \frac{1}{2}[A, B]_{\mathfrak{m}}$, for $A, B \in \mathfrak{m}$ (see proposition 6.62 in [13]). In fact, every normal homogenous space is naturally reductive (see page 220 in [13] or [2]).*

Example 4.4 *A example more general than above is the following. Let $M = G/H$ be a homogeneous space. We admit that M has a G-invariant metric $\ll, \gg$. Using Theorem 3.36 in [5] we obtain a left invariant metric $<, >$ on G such that $\pi : G \to G/H$ is a Riemannian submersion. Theorem 4.7 assures that every $\nabla^{G/H}$-martingale X_t is written as $X_t = \pi \circ e^G(M)$, where M is a local martingale in $\mathfrak{m}$.*

5 Martingales in sphere

Let S^n be a sphere n-dimensional in $\mathbb{R}^n$. We can write S^n as a normal homogeneous space in the following way. In [13], we found in Example 6.61(a) that if we define a bi-invariant metric on $SO(n+1)$ by $< U, V >= \frac{1}{2}\mathrm{tr}(U^t V) = -B(U, V)/(2n-2)$, $n \geq 2$, B is the Killing form, then $S^n = SO(n+1)/SO(n)$ is a normal homogeneous space. Furthermore, the normal homogenous metric on $S^n = SO(n+1)/SO(n)$ is the usual metric on S^n. It directly follows that $SO(n+1)/SO(n)$ is a reductive homogeneous space. The reductive decomposition is given by $\mathfrak{o}(n+1) = \mathfrak{o}(n) + \mathfrak{m}$, where $\mathfrak{m}$ is the subspace of all $n \times n$

matrices of the form
$$\begin{pmatrix} 0 & -x^t \\ x & 0_n \end{pmatrix},$$
where $x = (x_1, \ldots, x_n)$ is a column vector in $\mathbb{R}^n$ and 0_n the $n \times n$ zero matrix. It is clear that $\mathfrak{m}$ is isomorph to $\mathbb{R}^n$. Let us denote such isomorphism by $\phi : \mathfrak{m} \to \mathbb{R}^n$. It is immediate that a semimartingale ξ in $\mathbb{R}^n$ is a local martingale if and only if $\phi(\xi) = M$ is a local martingale in $\mathfrak{m}$.

Theorem 5.1 *Let S^n be a sphere n-dimensional in $\mathbb{R}^n$ with its usual metric induced of $\mathbb{R}^{n+1}$. There is a 1-1 correspondence between matingales in S^n and local martingales in $\mathbb{R}^n$.*

Proof: Let X_t be a ∇^{S^n}-martingale in S^n, where ∇^{S^n} is the Levi-Civita connection. Theorem 4.7 yields a unique local martingale in $\mathfrak{m}$ such that $X_t = \pi \circ e^G(M)$, where ∇^G is the Levi-Civita connection on $SO(n+1)$. Using the isomorphism $\phi : \mathbb{R}^n \to \mathfrak{m}$ defined above we see that $M = \phi(\xi)$, where ξ is the unique local martingale in $\mathbb{R}^n$ that satisfies such relation. It follows that X_t is unique related with ξ, and the proof is complete.

By Remark 2 we know that a ∇^{S^n}-martingale X_t satisfies the Itô stochastic differential equation
$$d^{S^n} X_t = \tau_{Y_t *} dM_t, X_0 = o^t,$$
where M_t is a local martingale in $\mathfrak{m}$ and $o^t = (1, 0, \ldots, 0)$. In the other hand, there exists a unique local martingale ξ such that $M = \phi(\xi)$. So, for a 1-form θ we can compute
$$\int \theta d^{S^n} X_t = \int \theta \tau_{Y_t *} dM_t = \int \theta \tau_{Y_t *} d\phi(\xi)_t = \int \theta \tau_{Y_t *} \phi_* d\xi_t,$$
where we used the geometric Itô formula (1) in the last equality. Thus X_t satisfies the following Itô differential equation
$$d^{S^n} X_t = \tau_{Y_t *} \phi_* d\xi_t, \xi_0 = (0, 0, \ldots, 0).$$

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